Optical Tunnelling
from One-Dimensional Square Well Potentials

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Abstract

We study a one-dimensional model for optical tunnelling with a refractive index in the shape of a square well. The relevance of the model and its limiting cases are discussed. The main result is the leading behaviour of the exponentially small imaginary part of the eigenvalue which determines the radiation loss. To calculate the leading behaviour of the imaginary part we use Berry's formula which controls the asymptotic expansion of the Airy function $Bi(z)$ to better-than-exponential accuracy.

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1 Introduction

In a recent edition of this Journal, Kath and Kriegsmann [5] have studied radiation losses in a weakly guiding optical fibre which is slightly bent. They derive the equation

$$\nabla^2 y + f(\xi,\eta)y - \mu y + \varepsilon\alpha y = 0, \tag{1.1}$$

which is central to their analysis. (In eq. (1.1), the ξ-axis and the η-axis are the coordinate axes orthogonal to the fibre in a torsion-free comoving coordinate system; $f(\xi,\eta)$ gives the refractive index in the core relative to the cladding; ε is a small positive number which measures the curvature of the fibre, and α is a function linear in ξ and η.) Equation (1.1), together with appropriate boundary conditions, constitutes an eigenvalue problem in the parameter μ, and the energy loss can be calculated from the imaginary part of the eigenvalue.

Using WKB techniques and asymptotic matching, Kath and Kriegsmann calculate Im μ. Because it is difficult to give a completely rigorous analysis of the relatively complicated equation (1.1), it is desirable to study first a simpler model problem which can be solved explicitly and understood in detail. This has been done by Paris and Wood in this Journal [7], who solved the eigenvalue problem in λ:

$$-y''(x) - \varepsilon x y(x) = \lambda y(x); \quad x \in (0,\infty), \tag{1.2}$$

$$y'(0) + hy(0) = 0, \tag{1.3}$$

with an appropriate condition for $x \to \infty$. (Here h is a positive constant).

Problem (1.2/3) can be considered to be the limit as $\delta \to 0$ of the following problem:

$$-y''(x) + V(x)y(x) = \lambda y(x); \quad x \in \mathbf{R}, \tag{1.4}$$

where

$$V(x) = \begin{cases} -\varepsilon|x| & (|x| > \delta/2) \\ -\frac{2h}{\delta} - \varepsilon|x| & (|x| \le \delta/2) \end{cases}, \tag{1.5}$$

and $y \in C^1(\mathbf{R})$. Reflection symmetry implies that the lowest eigenfunction is even (and its derivative is odd), so that we can restrict our attention to $x \in (0,\infty)$. In the limit as δ tends to zero, we obtain a delta function potential at $x = 0^+$ and the jump condition (1.3) for the derivative of y.

Problem (1.2/3) shares some typical features with problem (1.1) but clearly lacks others. Firstly, the optical tunnelling problem is not symmetric (radiation goes out to one side only); secondly, the approximation for a weakly guiding fibre is obviously very poor for a delta function potential; and thirdly, the optical tunnelling problem is a two-dimensional problem. In this paper, we will present a more realistic one-dimensional model to which the first two criticisms do not apply. In response to the third deficiency, the study of a one-dimensional model is justified by the physical fact that radiation is mainly in the plane of the bend. Thus we expect that a one-dimensional model will explain, to a great extent, the mechanism of radiation loss in a bent fibre.

2 The model

In the more realistic model the equation

$$-y''(x) + V(x)y(x) = \lambda y(x); \qquad x \in \mathbf{R}, \tag{2.1}$$

where

$$V(x) = \begin{cases} -\varepsilon x & (|x| > \delta/2) \\ -\frac{2h}{\delta} - \varepsilon x & (|x| \le \delta/2) \end{cases}, \tag{2.2}$$

($h > 0$) replaces eqs. (1.4/5). We require solutions $y \in C^1(\mathbf{R})$ to this equation for very small ε. δ is assumed to be much larger than ε, but otherwise arbitrary. In particular, δ may be small, so that the weakly guiding approximation is justified. To complete the model we must discuss the boundary conditions.

As $x \to +\infty$, the leading behaviours of solutions of (2.1/2) are linear combinations of the WKB approximations

$$f_{\pm}(x) = \pi^{-1/2}\varepsilon^{-1/12}x^{-1/4}e^{\pm i(2\varepsilon^{1/2}x^{3/2}/3+\pi/4)}, \tag{2.3}$$

which are interpreted as outgoing or incoming waves, respectively. (Whether we interpret f_+ or f_- as an outgoing wave is up to us. In the complete model, of course, the interpretation must follow from the underlying partial differential equations which contain the time dependence.) In case of optical tunnelling we have to choose the outgoing wave condition

$$y(x) \sim f_+(x) \quad \text{for} \quad x \to +\infty. \tag{2.4}$$

As $x \to -\infty$, the leading behaviours of solutions of (2.1) are linear combinations of the WKB approximations

$$g_{\pm}(x) = \frac{1}{2}\pi^{-1/2}\varepsilon^{-1/12}(-x)^{-1/4}e^{\pm 2\varepsilon^{1/2}(-x)^{3/2}/3}. \tag{2.5}$$

Obviously, we have to choose

$$y(x) \sim c g_-(x) \quad \text{for} \quad x \to -\infty, \tag{2.6}$$

where c is a contant. To find $C^1(\mathbf{R})$ solutions of (2.1/2) which satisfy (2.4) and (2.6) is our model problem.

It is easy to see that the problem (2.1/2), (2.4), and (2.6) can be reformulated as the following eigenvalue problem: Find the eigenvalue $\lambda \in \mathbf{C}$ for which the equation (2.1) has a solution that is square-integrable along rays $\{xe^{i\theta} | x \in \mathbf{R}, \theta \text{ fixed, small and positive}\}$. In this form, the problem fits into the mathematical theory of resonances (see, for example, ref. [4] and references therein), which has been developed to give a mathematically rigorous description of resonances in quantum mechanics. The potential (2.2), however, is not among the class of potentials discussed in these papers and we have to find the eigenvalues by different means. Before we determine the eigenvalues to leading orders in the next section, however, we discuss here the limiting cases $\varepsilon = 0$ and $\delta \to 0$.

For $\varepsilon = 0$, we are looking for square-integrable solutions of (2.1), which describe bound states in quantum mechanics. It is well-known (see for example ref. [3]), that the eigensolutions are even, $y_+(-x) = y_+(x)$, with

$$y_+(x) = \begin{cases} \cos\left(\sqrt{\frac{2h}{\delta}+\lambda}\frac{\delta}{2}\right)e^{\sqrt{-\lambda}(\frac{\delta}{2}-x)} & (x > \delta/2) \\ \cos\left(\sqrt{\frac{2h}{\delta}+\lambda}x\right) & (0 < x \leq \delta/2), \end{cases} \tag{2.7}$$

or odd, $y_-(x) = -y_-(x)$, with

$$y_-(x) = \begin{cases} \sin\left(\sqrt{\frac{2h}{\delta}+\lambda}\frac{\delta}{2}\right)e^{\sqrt{-\lambda}(\frac{\delta}{2}-x)} & (x > \delta/2) \\ \sin\left(\sqrt{\frac{2h}{\delta}+\lambda}x\right) & (0 < x \leq \delta/2). \end{cases} \tag{2.8}$$

The eigenvalues are given by the conditions

$$\sqrt{-\lambda}\cot\left(\sqrt{\frac{2h}{\delta}+\lambda}\frac{\delta}{2}\right) = \sqrt{\frac{2h}{\delta}+\lambda}, \tag{2.9}$$

$$-\sqrt{-\lambda}\tan\left(\sqrt{\frac{2h}{\delta}+\lambda}\frac{\delta}{2}\right) = \sqrt{\frac{2h}{\delta}+\lambda}, \tag{2.10}$$

respectively. For $\delta << 1$, eq. (2.9) has only one solution, namely,

$$\lambda = -h^2 + \frac{2}{3}h^3\delta + 0(\delta^2). \tag{2.11}$$

There are no odd eigenfunctions for $\delta << 1$.

For small but nonzero ε and $\delta \to 0$, we expect our model to become

$$-y''(x) - \varepsilon x y(x) = \lambda y(x); \qquad x \in \mathbf{R}\backslash\{0\}, \tag{2.12}$$

$$y'(0^+) - y'(0^-) + 2hy(0) = 0, \tag{2.13}$$

$$y \in C^0(\mathbf{R}), \tag{2.14}$$

where the jump condition (2.13) arises from the delta function potential V. The solutions of (2.12) which satisfy the boundary conditions (2.4) and (2.6) are

$$x \geq 0 : y(x) = i \; Ai(-\varepsilon^{-2/3}\lambda - \varepsilon^{1/3}x) + Bi(-\varepsilon^{-2/3}\lambda - \varepsilon^{1/3}x), \tag{2.15}$$

$$x \leq 0 : y(x) = c \; Ai(-\varepsilon^{-2/3}\lambda - \varepsilon^{1/3}x). \tag{2.16}$$

The conditions (2.13) and (2.14) lead to the equation

$$\begin{aligned} &Ai'(-\varepsilon^{-2/3}\lambda)Bi(-\varepsilon^{-2/3}\lambda) - Ai(-\varepsilon^{-2/3}\lambda)Bi'(-\varepsilon^{-2/3}\lambda) \\ &+ 2\varepsilon^{-1/3}h[i \; Ai(-\varepsilon^{-2/3}\lambda)Ai(-\varepsilon^{-2/3}\lambda) + Ai(-\varepsilon^{-2/3}\lambda)Bi(-\varepsilon^{-2/3}\lambda)] = 0 \end{aligned} \tag{2.17}$$

for the eigenvalue $\lambda = \kappa + i\nu$.

We now look for a solution of eq. (2.17) with negative κ of order $0(1)$ and negative ν of order $0(\exp[-4(-\kappa)^{3/2}/(3\varepsilon)])$ (which indeed we will find). To solve the equation we split into real and imaginary parts, use the asymptotic expansions (4.1)-(4.4) and the fact that up to the order required,

$$\begin{aligned} &\mathrm{Im} Ai(-\varepsilon^{-2/3}\lambda) \sim -\varepsilon^{-2/3}\nu Ai'(-\varepsilon^{-2/3}\kappa), \\ &\mathrm{Im} Ai'(-\varepsilon^{-2/3}\lambda) \sim \varepsilon^{-4/3}\kappa\nu Ai(-\varepsilon^{-2/3}\kappa), \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\mathrm{Im}\; Bi(-\varepsilon^{-2/3}\lambda) \sim -\varepsilon^{-2/3}\nu Bi'(-\varepsilon^{-2/3}\kappa), \\ &\mathrm{Im}\; Bi'(-\varepsilon^{-2/3}\lambda) \sim \varepsilon^{-4/3}\kappa\nu Bi(-\varepsilon^{-2/3}\kappa). \end{aligned} \tag{2.19}$$

Formulae (2.19) require justification. Since Im $Bi(-\varepsilon^{-2/3}\lambda)$ is of order $0(\varepsilon^{-5/6}\nu \exp[2(-\kappa)^{3/2}/(3\varepsilon)])$, we have to control the asymptotic expansion of the Airy function $Bi(z)$ to better-than-exponential accuracy. This has been achieved by Berry [2] (with a rigorous underpinning given by Olver [6]) who derives the formula

$$\begin{aligned} &\mathrm{Im}\left\{\pi^{1/2}\left(\frac{3}{4}F\right)^{1/6} e^{F/2} Bi\left[\left(\frac{3}{4}F\right)^{2/3}\right]\right. \\ &\left. - e^F \sum_{r=0}^{\mathrm{Int}[|F|+\alpha]} (r - \frac{1}{6})!(r - \frac{5}{6})/(2\pi r! F^r)\right\} \to \frac{1}{\sqrt{\pi}}\int_0^\sigma dt\; e^{-t^2} \quad \text{as } |F| \to \infty, \end{aligned} \tag{2.20}$$

where $F = 4(-\varepsilon^{-2/3}\lambda)^{3/2}/3$ and $\sigma \sim -\sqrt{3/(2\varepsilon)}\nu/(-x)^{1/4}$, and where α is of order one. To leading order in ε, σ is zero and therefore the $e^{-2(-\varepsilon^{-2/3}\lambda)^{3/2}/3}$ term does not contribute to Im $Bi(-\varepsilon^{-2/3}\lambda)$ nor to Im $Bi'(-\varepsilon^{-2/3}\lambda)$, up to order $0(\varepsilon^{1/6}\exp[-2(-\kappa)^{3/2}/(3\varepsilon)])$, which is the accuracy required.

Splitting real and imaginary part in eq. (2.17) we find that the real part to leading order, is given by

$$Ai'(-\varepsilon^{-2/3}\kappa)Bi(-\varepsilon^{-2/3}\kappa) - Ai(-\varepsilon^{-2/3}\kappa)Bi'(-\varepsilon^{-2/3}\kappa)$$

$$+ 2\varepsilon^{-1/3}h\ Ai(-\varepsilon^{-2/3}\kappa)Bi(-\varepsilon^{-2/3}\kappa) = 0. \tag{2.21}$$

Using the asymptotic expansions (4.1)-(4.4) yields

$$(-\kappa)^{1/2} \sim h\left[1 + \varepsilon^2\frac{5}{32}(-\kappa)^{-3}\right], \tag{2.22}$$

and therefore, for $\kappa \sim \kappa_0 + \varepsilon\kappa_1 + \varepsilon^2\kappa_2 + \cdots$,

$$\kappa \sim -h^2 - \varepsilon^2\frac{5}{16h^4} + 0(\varepsilon^3), \tag{2.23}$$

as $\varepsilon \to 0$.

To leading order, the imaginary part of eq. (2.17) is given by

$$\varepsilon^{-2/3}\nu[Ai'(-\varepsilon^{-2/3}\kappa)Bi(-\varepsilon^{-2/3}\kappa) + Ai(-\varepsilon^{-2/3}\kappa)Bi'(-\varepsilon^{-2/3}\kappa)] = [Ai(-\varepsilon^{-2/3}\kappa)]^2. \tag{2.24}$$

From this equation and (4.1)-(4.4) we find the leading term in the expansion of ν:

$$\nu \sim -h^2 \exp\left[-\frac{4}{3\varepsilon}h^3\right] \tag{2.25}$$

as $\varepsilon \to 0$. This differs from the result in model (1.2/3) by a factor $2/e$ in ν. That a factor 2 appears is not surprising because physically, in model (1.2/3), an equal amount of radiation goes out to both sides. The factor e^{-1} appears because of the linear term in the ε-expansion of κ for the model (1.2/3). In the model (2.12/13/14) the linear term in the ε-expansion of κ vanishes.

3 The imaginary part of the eigenvalue

We now return to the main objective of this paper and study the model (2.1/2) with the boundary conditions (2.4) and (2.6). The solutions are

$$x > \frac{\delta}{2} : y = i\ Ai(-\varepsilon^{-2/3}\lambda - \varepsilon^{1/3}x) + Bi(-\varepsilon^{-2/3}\lambda - \varepsilon^{1/3}x), \tag{3.1}$$

$$|x| \le \frac{\delta}{2} : y = c_1 Ai(-\varepsilon^{-2/3}(\frac{2h}{\delta} + \lambda) - \varepsilon^{1/3}x) + c_2 Bi(-\varepsilon^{-2/3}(\frac{2h}{\delta} + \lambda) - \varepsilon^{1/3}x), \tag{3.2}$$

$$x < -\frac{\delta}{2} : y = c\ Ai(-\varepsilon^{-2/3}\lambda - \varepsilon^{1/3}x). \tag{3.3}$$

Matching these solutions and their derivatives at $x = \pm\delta/2$ gives the condition for the eigenvalues. If we introduce the notation

$$\begin{aligned}
Ai\left[x \to -\frac{\delta}{2}^+\right] &= Ai\left(-\varepsilon^{-2/3}\left(\frac{2h}{\delta} + \lambda\right) + \varepsilon^{1/3}\frac{\delta}{2}\right), \\
Ai\left[x \to -\frac{\delta}{2}^-\right] &= Ai\left(-\varepsilon^{-2/3}\lambda + \varepsilon^{1/3}\frac{\delta}{2}\right), \\
Ai\left[x \to \frac{\delta}{2}^+\right] &= Ai\left(-\varepsilon^{-2/3}\lambda - \varepsilon^{1/3}\frac{\delta}{2}\right), \\
Ai\left[x \to \frac{\delta}{2}^-\right] &= Ai\left(-\varepsilon^{-2/3}\left(\frac{2h}{\delta} + \lambda\right) - \varepsilon^{1/3}\frac{\delta}{2}\right),
\end{aligned} \tag{3.4}$$

and analogous formulae for Ai', Bi' and Bi, the eigenvalue condition reads

$$\begin{vmatrix} Ai\left[x \to -\frac{\delta^+}{2}\right] & Bi\left[x \to -\frac{\delta^+}{2}\right] & Ai\left[x \to -\frac{\delta^-}{2}\right] & 0 \\ Ai'\left[x \to -\frac{\delta^+}{2}\right] & Bi'\left[x \to -\frac{\delta^+}{2}\right] & Ai'\left[x \to -\frac{\delta^-}{2}\right] & 0 \\ Ai\left[x \to \frac{\delta^-}{2}\right] & Bi\left[x \to \frac{\delta^-}{2}\right] & 0 & i\,Ai\left[x \to \frac{\delta^+}{2}\right] + Bi\left[x \to \frac{\delta^+}{2}\right] \\ Ai'\left[x \to \frac{\delta^-}{2}\right] & Bi'\left[x \to \frac{\delta^-}{2}\right] & 0 & i\,Ai'\left[x \to \frac{\delta^+}{2}\right] + Bi'\left[x \to \frac{\delta^+}{2}\right] \end{vmatrix} = 0 \qquad (3.5)$$

To evaluate the eigenvalue condition, we again split real and imaginary parts for which, as in section 2, we use

$$\begin{aligned} Ai(-\varepsilon^{-2/3}\lambda + \varepsilon^{1/3}\frac{\delta}{2}) &\sim Ai(-\varepsilon^{-2/3}\kappa + \varepsilon^{1/3}\frac{\delta}{2}) \\ &- i\varepsilon^{-2/3}\nu Ai'(-\varepsilon^{-2/3}\kappa + \varepsilon^{1/3}\frac{\delta}{2}), \end{aligned} \qquad (3.6)$$

and analogous formulae for Ai', Bi, Bi' for the appropriate arguments. To derive the formulae for Ai' and Bi' we use Airy's equation $y''(z) = zy(z)$. To show that the formulae for $Bi(z)$ and $Bi'(z)$ for large z near the positive real axis give indeed the correct imaginary parts we again have to use Berry's formula [2]. Next we expand $\kappa \sim \kappa_0 + \varepsilon\kappa_1 + \varepsilon^2\kappa_2 + 0(\varepsilon^3)$ as $\varepsilon \to 0$ and use the asymptotic expansions (4.7)-(4.14). If we then set the real part of the determinant in eq. (3.5) to leading order to zero we obtain

$$\begin{aligned} &\begin{vmatrix} S\left(-\frac{\delta}{2}\right) & C\left(-\frac{\delta}{2}\right) & 1 & 0 \\ -\sqrt{\frac{2h}{\delta}+\kappa_0}\,C\left(-\frac{\delta}{2}\right) & \sqrt{\frac{2h}{\delta}+\kappa_0}\,S\left(-\frac{\delta}{2}\right) & -\sqrt{-\kappa_0} & 0 \\ S\left(\frac{\delta}{2}\right) & C\left(\frac{\delta}{2}\right) & 0 & 1 \\ -\sqrt{\frac{2h}{\delta}+\kappa_0}\,C\left(\frac{\delta}{2}\right) & \sqrt{\frac{2h}{\delta}+\kappa_0}\,S\left(\frac{\delta}{2}\right) & 0 & \sqrt{-\kappa_0} \end{vmatrix} \\ &= \left(\frac{2h}{\delta} + 2\kappa_0\right)\left[C\left(-\frac{\delta}{2}\right)S\left(\frac{\delta}{2}\right) - C\left(\frac{\delta}{2}\right)S\left(-\frac{\delta}{2}\right)\right] \\ &- 2\sqrt{-\kappa_0}\sqrt{\frac{2h}{\delta}+\kappa_0}\left[C\left(-\frac{\delta}{2}\right)C\left(\frac{\delta}{2}\right) + S\left(-\frac{\delta}{2}\right)S\left(\frac{\delta}{2}\right)\right] \\ &= 2\left[\sqrt{\frac{2h}{\delta}+\kappa_0}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\frac{\delta}{2}\right) - \sqrt{-\kappa_0}\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\frac{\delta}{2}\right)\right] \\ &\times \left[\sqrt{\frac{2h}{\delta}+\kappa_0}\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\frac{\delta}{2}\right) + \sqrt{-\kappa_0}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\frac{\delta}{2}\right)\right] = 0, \end{aligned} \qquad (3.7)$$

where we have used the notation

$$S(x) = \sin\left[\frac{2}{3\varepsilon}\left(\frac{2h}{\delta} + \kappa_0 + \varepsilon\kappa_1\right)^{3/2} + \frac{\pi}{4} + \sqrt{\frac{2h}{\delta} + \kappa_0}\,x\right], \qquad (3.8)$$

$$C(x) = \cos\left[\frac{2}{3\varepsilon}\left(\frac{2h}{\delta} + \kappa_0 + \varepsilon\kappa_1\right)^{3/2} + \frac{\pi}{4} + \sqrt{\frac{2h}{\delta} + \kappa_0}\,x\right]. \qquad (3.9)$$

Our asymptotic expansions are written in terms of $S(x)$ and $C(x)$ because they are the linear combinations of $\sin(\sqrt{\frac{2h}{\delta}+\kappa_0 x})$ and $\cos(\sqrt{\frac{2h}{\delta}+\kappa_0 x})$ which give the correct solution for $\varepsilon = 0$. Equation (3.7), as expected, gives the eigenvalue conditions we found in the case $\varepsilon = 0$; see eqs. (2.9/10).

To find κ_1, we have to evaluate the real part of eq. (3.5) to next order. The result of this calculation is denoted in terms of D_i^j, which is defined to be the derminant one obtains if the jth ε-term of the appropriate expansion in (4.7)-(4.14) is inserted into the ith column of the determinant in (3.5). Each of these determinants D_i^j is, except for multiplicative factors, of the same type as the determinant in (3.7). The result of the calculation is as follows:

$$
\begin{aligned}
D_1^1 + D_2^1 &= 0, &(3.10)\\
D_1^2 + D_2^2 &= 0, &(3.11)\\
D_1^3 + D_2^3 &= \frac{1}{2}\varepsilon\pi^{-2}\kappa_1\left(\frac{2h}{\delta}+\kappa_0\right)^{-3/2} e^{-\sqrt{-\kappa_0}\delta}\\
&\times \left[\sqrt{-\kappa_0}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0\delta}\right)+\sqrt{\frac{2h}{\delta}+\kappa_0}\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0\delta}\right)\right], &(3.12)\\
D_1^4 + D_2^4 &= \frac{\varepsilon}{4}\pi^{-2}\kappa_1\delta\left(\frac{2h}{\delta}+\kappa_0\right)^{-1}(-\kappa_0)^{-1/2}e^{-\sqrt{-\kappa_0}\delta}\\
&\times \left[\left(\frac{2h}{\delta}+2\kappa_0\right)\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0\delta}\right)+2\sqrt{-\kappa_0}\sqrt{\frac{2h}{\delta}+\kappa_0}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\right)\right], &(3.13)\\
D_1^5 = D_2^5 &= D_1^6 = D_2^6 = 0, &(3.14)\\
D_3^1 + D_4^1 &= 0, &(3.15)\\
D_3^2 + D_4^2 &= 0, &(3.16)\\
D_3^3 + D_4^3 &= -\frac{\varepsilon}{2}\pi^{-2}\kappa_1(-\kappa_0)^{-3/2}e^{-\sqrt{-\kappa_0}\delta}\\
&\times \left[\sqrt{-\kappa_0}\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0\delta}\right)-\sqrt{\frac{2h}{\delta}+\kappa_0}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0\delta}\right)\right], &(3.17)\\
D_3^4 = D_4^4 &= D_3^5 = D_4^5 = D_3^6 = D_4^6 = 0. &(3.18)
\end{aligned}
$$

In some of the calculations eq. (3.7) has been used. The results show that $\kappa_1 = 0$ to all orders in δ and not only for $\delta \to 0$, which was the result given in eq. (2.22). This means that the κ_1-terms in (4.7)-(4.14) do not contribute to the calculation of ν.

In the calculation of the imaginary part ν of λ, we obtain contributions from the $i\ Ai\left[x \to \frac{\delta}{2}^+\right]$ and the $i\ Ai'\left[x \to \frac{\delta}{2}^+\right]$ terms in the fourth column of (3.5). If we take the lowest order terms in the other columns we obtain the determinant

$$
\begin{aligned}
D &= -\frac{i}{2}\pi^{-2}(-\kappa_0)^{-1/2}\exp\left[-\frac{4}{3\varepsilon}(-\kappa_0)^{3/2}\right]\\
&\times \left[\sqrt{-\kappa_0}\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0\delta}\right)-\sqrt{\frac{2h}{\delta}+\kappa_0}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0\delta}\right)\right]. \qquad (3.19)
\end{aligned}
$$

The extra contributions come from the other terms in (3.5) through the use of (3.6) and the corresponding formulae for the other Airy function and its derivative.

If we use these same formulae on column k in (3.5) and retain only the lowest order terms, all four possible determinants are zero. We have to go to next order, use (3.6) and its associated formulae in column k, and insert the jth ε-term into column i. The resulting determinant is denoted by D_i^{jk}. The non-zero results are

$$\begin{aligned} D_1^{11}+D_2^{11}+D_1^{12} &+ D_2^{12} \\ &= -\frac{i}{2}\nu\pi^{-2}h\left(\frac{2h}{\delta}+\kappa_0\right)^{-1}(-\kappa_0)^{-1/2}\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\delta\right)e^{-\sqrt{-\kappa_0}\delta}, \end{aligned} \tag{3.20}$$

$$\begin{aligned} D_1^{21}+D_2^{21}+D_1^{22} &+ D_2^{22} \\ &= \frac{i}{2}\nu\pi^{-2}\frac{h}{\delta}\left(\frac{2h}{\delta}+\kappa_0\right)^{-3/2}(-\kappa_0)^{-1/2}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\delta\right)e^{-\sqrt{-\kappa_0}\delta}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} D_3^{13}+D_4^{13}+D_3^{14} &+ D_4^{14} = -\frac{i}{2}\nu\pi^{-2}\delta(-\kappa_0)^{-1}e^{-\sqrt{-\kappa_0}\delta} \\ &\times \left[\sqrt{-\kappa_0}\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\delta\right)-\sqrt{\frac{2h}{\delta}+\kappa_0}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\delta\right)\right], \end{aligned} \tag{3.22}$$

$$\begin{aligned} D_3^{23}+D_4^{23}+D_3^{24} &+ D_4^{24} = -\frac{i}{2}\nu\pi^{-2}(-\kappa_0)^{-3/2}e^{-\sqrt{-\kappa_0}\delta} \\ &\times \left[\sqrt{-\kappa_0}\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\delta\right)-\sqrt{\frac{2h}{\delta}+\kappa_0}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\delta\right)\right]. \end{aligned} \tag{3.23}$$

There are other terms which still have to be considered. These terms stem from the formula for the second derivative

$$Ai''\left(-\varepsilon^{-2/3}\left(\frac{2h}{\delta}+\kappa_0+\varepsilon x+\cdots\right)\right)=-\varepsilon^{-2/3}\left(\frac{2h}{\delta}+\kappa_0+\varepsilon x+\cdots\right)Ai\left(-\varepsilon^{-2/3}\left(\frac{2h}{\delta}+\kappa_0+\varepsilon x+\cdots\right)\right) \tag{3.24}$$

and similar formulae for $Bi''(-\varepsilon^{-2/3}(2h/\delta+\kappa_0+\varepsilon x+\cdots)), Ai''(-\varepsilon^{-2/3}(\kappa_0+\varepsilon x+\cdots)$ and $Bi''(-\varepsilon^{-2/3}(\kappa_0+\varepsilon x+\cdots))$. The εx-term in the factors in front of Ai or Bi lead to other determinants which have not yet been considered. It may be shown, however, that the resulting determinants sum to zero.

Adding up the determinants (3.19) and (3.20)-(3.23) and setting the result equal to zero yields an equation linear in ν which may easily be solved to obtain for small ε

$$\begin{aligned} \nu \sim -\Big\{&(-\kappa_0)^{-1/2}[\delta+(-\kappa_0)^{-1/2}] \\ &+h\left(\frac{2h}{\delta}+\kappa_0\right)^{-3/2}\frac{\sqrt{\frac{2h}{\delta}+\kappa_0}\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\delta\right)-\frac{1}{\delta}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\delta\right)}{\sqrt{-\kappa_0}\cos\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\delta\right)-\sqrt{\frac{2h}{\delta}+\kappa_0}\sin\left(\sqrt{\frac{2h}{\delta}+\kappa_0}\delta\right)}\Big\}^{-1} \\ &\times\exp\left[-\frac{4}{3\varepsilon}(-\kappa_0)^{3/2}+\sqrt{-\kappa_0}\delta\right]. \end{aligned} \tag{3.25}$$

The important feature is the exponentially small factor $\exp[-4h^3/(3\varepsilon)]$. In the limit $\delta \to 0$, (3.25), of course, reduces to (2.25).

Our results can be generalized to potentials of the form

$$V(x) = \begin{cases} -\varepsilon x & (|x| > a) \\ -\rho_i - \varepsilon x & (x_i < x < x_{i+1}, i = 1, 2, \cdots n) \end{cases} \tag{3.26}$$

where the interval $[-a, a]$ has been partitioned, $-a = x_1 < x_2 < \cdots < x_{n+1} = a$, with $\varepsilon << \rho_i (i = 1, 2, \cdots, n)$. Using these potentials, we can approximate the shape of any given potential. Because of the complexity of the calculation, to find ν one has to use a symbolic computation program such as MACSYMA.

4 Appendix: Asymptotic Expansions

Using the well-known asymptotic expansions for Airy functions given in [1], we list here the asymptotic expansions for the necessary arguments and up to the order required. For negative κ of order 0(1) and small positive ε, we need in section 2 the following expansions:

$$\begin{aligned} Ai(-\varepsilon^{-2/3}\kappa) &\sim \frac{1}{2}\pi^{-1/2}\varepsilon^{1/6}(-\kappa)^{-1/4}\exp\left[-\frac{2}{3\varepsilon}(-\kappa)^{3/2}\right] \\ &\times \left[1 - \varepsilon\frac{5}{48}(-\kappa)^{-3/2} + \varepsilon^2\frac{385}{4608}(-\kappa)^{-3} + 0(\varepsilon^3)\right], \end{aligned} \tag{4.1}$$

$$\begin{aligned} Ai'(-\varepsilon^{-2/3}\kappa) &\sim -\frac{1}{2}\pi^{-1/2}\varepsilon^{-1/6}(-\kappa)^{1/4}\exp\left[-\frac{2}{3\varepsilon}(-\kappa)^{3/2}\right] \\ &\times \left[1 + \varepsilon\frac{7}{48}(-\kappa)^{-3/2} - \varepsilon^2\frac{455}{4608}(-\kappa)^{-3} + 0(\varepsilon^3)\right], \end{aligned} \tag{4.2}$$

$$\begin{aligned} Bi(-\varepsilon^{-2/3}\kappa) &\sim \pi^{-1/2}\varepsilon^{1/6}(-\kappa)^{-1/4}\exp\left[\frac{2}{3\varepsilon}(-\kappa)^{3/2}\right] \\ &\times \left[1 + \varepsilon\frac{5}{48}(-\kappa)^{-3/2} + \varepsilon^2\frac{385}{4608}(-\kappa)^{-3} + 0(\varepsilon^3)\right], \end{aligned} \tag{4.3}$$

$$\begin{aligned} Bi'(-\varepsilon^{-2/3}\kappa) &\sim \pi^{-1/2}\varepsilon^{-1/6}(-\kappa)^{1/4}\exp\left[\frac{2}{3\varepsilon}(-\kappa)^{3/2}\right] \\ &\times \left[1 - \varepsilon\frac{7}{48}(-\kappa)^{-3/2} - \varepsilon^2\frac{455}{4608}(-\kappa)^{-3} + 0(\varepsilon^3)\right]. \end{aligned} \tag{4.4}$$

Furthermore, for negative ν of order $0(\exp[-4(-\kappa_0)^{3/2}/(3\varepsilon)])$ up to the order required, Re $Ai(-\varepsilon^{-2/3}\lambda) \sim Ai(-\varepsilon^{-2/3}\kappa)$, Re $Ai'(-\varepsilon^{-2/3}\lambda) \sim Ai'(-\varepsilon^{-2/3}\kappa)$, Re $Bi(-\varepsilon^{-2/3}\lambda) \sim Bi(-\varepsilon^{-2/3}\kappa)$, and Re $Bi'(-\varepsilon^{-2/3}\lambda) \sim Bi(-\varepsilon^{-2/3}\kappa)$.

In section 3 we require certain asymptotic expansions which we write in terms of

$$S(x) = \sin\left[\frac{2}{3\varepsilon}\left(\frac{2h}{\delta} + \kappa_0 + \varepsilon\kappa_1\right)^{3/2} + \frac{\pi}{4} + \sqrt{\frac{2h}{\delta} + \kappa_0 x}\right], \tag{4.5}$$

$$C(x) = \cos\left[\frac{2}{3\varepsilon}\left(\frac{2h}{\delta} + \kappa_0 + \varepsilon\kappa_1\right)^{3/2} + \frac{\pi}{4} + \sqrt{\frac{2h}{\delta} + \kappa_0 x}\right]. \tag{4.6}$$

The relevant expansions are

$$
\begin{aligned}
& Ai\left[-\varepsilon^{-2/3}\left(\frac{2h}{\delta}+\kappa_0+\varepsilon\kappa_1+\varepsilon x+\varepsilon^2\kappa_2+\cdots\right)\right] \\
\sim\ & \pi^{-1/2}\varepsilon^{1/6}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1/4}\left\{S(x)+\varepsilon\left[-\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1}xS(x)\right.\right. \\
-\ & \frac{5}{48}\left(\frac{2h}{\delta}+\kappa_0\right)^{-3/2}C(x)-\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1}\kappa_1 S(x) \\
+\ & \left.\left.\frac{1}{2}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1/2}\kappa_1 xC(x)+\left(\frac{2h}{\delta}+\kappa_0\right)^{1/2}\kappa_2 C(x)+\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1/2}x^2C(x)\right]+\cdots\right\}, \quad (4.7)
\end{aligned}
$$

$$
\begin{aligned}
& Ai'\left[-\varepsilon^{-2/3}\left(\frac{2h}{\delta}+\kappa_0+\varepsilon\kappa_1+\varepsilon x+\varepsilon^2\kappa_2+\cdots\right)\right] \\
\sim\ & -\pi^{-1/2}\varepsilon^{-1/6}\left(\frac{2h}{\delta}+\kappa_0\right)^{1/4}\left\{C(x)+\varepsilon\left[\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1}xC(x)\right.\right. \\
-\ & \frac{7}{48}\left(\frac{2h}{\delta}+\kappa_0\right)^{-3/2}S(x)+\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1}\kappa_1 C(x) \\
-\ & \left.\left.\frac{1}{2}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1/2}\kappa_1 xS(x)-\left(\frac{2h}{\delta}+\kappa_0\right)^{1/2}\kappa_2 S(x)-\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1/2}x^2S(x)\right]+\cdots\right\}, \quad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
& Bi\left[-\varepsilon^{-2/3}\left(\frac{2h}{\delta}+\kappa_0+\varepsilon\kappa_1+\varepsilon x+\varepsilon^2\kappa_2+\cdots\right)\right] \\
\sim\ & \pi^{-1/2}\varepsilon^{1/6}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1/4}\left\{C(x)+\varepsilon\left[-\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1}xC(x)\right.\right. \\
+\ & \frac{5}{48}\left(\frac{2h}{\delta}+\kappa_0\right)^{-3/2}S(x)-\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1}\kappa_1 C(x) \\
-\ & \left.\left.\frac{1}{2}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1/2}\kappa_1 xS(x)-\left(\frac{2h}{\delta}+\kappa_0\right)^{1/2}\kappa_2 S(x)-\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1/2}x^2S(x)\right]+\cdots\right\}, \quad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
& Bi'\left[-\varepsilon^{-2/3}\left(\frac{2h}{\delta}+\kappa_0+\varepsilon\kappa_1+\varepsilon x+\varepsilon^2\kappa_2+\cdots\right)\right] \\
\sim\ & \pi^{-1/2}\varepsilon^{-1/6}\left(\frac{2h}{\delta}+\kappa_0\right)^{1/4}\left\{S(x)+\varepsilon\left[\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1}xS(x)\right.\right. \\
+\ & \frac{7}{48}\left(\frac{2h}{\delta}+\kappa_0\right)^{-3/2}C(x)+\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1}\kappa_1 S(x) \\
+\ & \left.\left.\frac{1}{2}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1/2}\kappa_1 xC(x)+\left(\frac{2h}{\delta}+\kappa_0\right)^{1/2}\kappa_2 C(x)+\frac{1}{4}\left(\frac{2h}{\delta}+\kappa_0\right)^{-1/2}x^2C(x)\right]+\cdots\right\}, \quad (4.10)
\end{aligned}
$$

$$
\begin{aligned}
& Ai[-\varepsilon^{-2/3}(\kappa_0+\varepsilon\kappa_1+\varepsilon x+\varepsilon^2\kappa_2+\cdots)] \\
\sim\ & \frac{1}{2}\pi^{-1/2}\varepsilon^{1/6}(-\kappa_0)^{-1/4}\exp\left[-\frac{2}{3\varepsilon}(-\kappa_0-\varepsilon\kappa_1)^{3/2}+(-\kappa_0)^{1/2}x\right] \\
\times\ & \left\{1+\varepsilon\left[-\frac{x}{4\kappa_0}-\frac{5}{48}(-\kappa_0)^{-3/2}-\frac{\kappa_1}{4\kappa_0}-\frac{1}{2}(-\kappa_0)^{-1/2}\kappa_1 x\right.\right. \\
+\ & \left.\left.(-\kappa_0)^{1/2}\kappa_2+\frac{1}{4}(-\kappa_0)^{-1/2}x^2\right]+\cdots\right\}, \quad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
& Ai'[-\varepsilon^{-2/3}(\kappa_0+\varepsilon\kappa_1+\varepsilon x+\varepsilon^2\kappa_2+\cdots)] \\
\sim\ & -\frac{1}{2}\pi^{-1/2}\varepsilon^{-1/6}(-\kappa_0)^{1/4}\exp\left[-\frac{2}{3\varepsilon}(-\kappa_0-\varepsilon\kappa_1)^{3/2}+(-\kappa_0)^{1/2}x\right] \\
\times\ & \left\{1+\varepsilon\left[\frac{x}{4\kappa_0}+\frac{7}{48}(-\kappa_0)^{-3/2}+\frac{\kappa_1}{4\kappa_0}-\frac{1}{2}(-\kappa_0)^{-1/2}\kappa_1 x\right.\right. \\
+\ & \left.\left.(-\kappa_0)^{1/2}\kappa_2+\frac{1}{4}(-\kappa_0)^{-1/2}x^2\right]+\cdots\right\},
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
& Bi[-\varepsilon^{-2/3}(\kappa_0+\varepsilon\kappa_1+\varepsilon x+\varepsilon^2\kappa_2+\cdots)] \\
\sim\ & \pi^{-1/2}\varepsilon^{1/6}(-\kappa_0)^{-1/4}\exp\left[\frac{2}{3\varepsilon}(-\kappa_0-\varepsilon\kappa_1)^{3/2}-(-\kappa_0)^{1/2}x\right] \\
\times\ & \left\{1+\varepsilon\left[-\frac{x}{4\kappa_0}+\frac{5}{48}(-\kappa_0)^{-3/2}-\frac{\kappa_1}{4\kappa_0}+\frac{1}{2}(-\kappa_0)^{1/2}\kappa_1 x\right.\right. \\
-\ & \left.\left.(-\kappa_0)^{1/2}\kappa_2-\frac{1}{4}(-\kappa_0)^{-1/2}x^2\right]+\cdots\right\},
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
& Bi'[-\varepsilon^{-2/3}(\kappa_0+\varepsilon\kappa_1+\varepsilon x+\varepsilon^2\kappa_2+\cdots)] \\
\sim\ & \pi^{-1/2}\varepsilon^{-1/6}(-\kappa_0)^{1/4}\exp\left[\frac{2}{3\varepsilon}(-\kappa_0-\varepsilon\kappa_1)^{3/2}-(-\kappa_0)^{1/2}x\right] \\
\times\ & \left\{1+\varepsilon\left[\frac{x}{4\kappa_0}-\frac{7}{48}(-\kappa_0)^{-3/2}+\frac{\kappa_1}{4\kappa_0}+\frac{1}{2}(-\kappa_0)^{-1/2}\kappa_1 x\right.\right. \\
-\ & \left.\left.(-\kappa_0)^{1/2}\kappa_2-\frac{1}{4}(-\kappa_0)^{-1/2}x^2\right]+\cdots\right\},
\end{aligned}
\tag{4.14}
$$

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